

SURREAL CANONICAL LINEAR ORDERS

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1. SUMMARY

In this note, we'll define a linearly ordered set S_d for any ordinal d so that any linearly ordered set X of cardinality \aleph_d can be embedded in S_d in an order-preserving manner. If one accepts the generalized continuum hypothesis, which states that any cardinals κ, λ cannot have $\lambda < \kappa < 2^\lambda$, then $\|S_d\| = \aleph_d$, and clearly the S_d are of optimal size to act as a universal receiver of other same-sized linear orders. The universal embedding result (theorem 2) uses the axiom of choice.

2. DEFINITIONS

We choose the von Neumann definition of *ordinal numbers*; informally, an ordinal d is the set of all previous ordinals. More formally, a set A is **transitive** iff $x \in A \ \& \ y \in x \implies y \in A$ (we'll use **bold** to indicate when a term is being defined). A set d is an **ordinal number** (or just an "ordinal"), written $d \in \mathcal{O}$ iff it is transitive and all its elements are ordinal numbers. It is a result that any set of ordinal numbers is well ordered by inclusion; this is what we mean when we write $x < y$ for $x, y \in \mathcal{O}$. It can be shown that every ordinal is of the form $\{d \in \mathcal{O} : d < e\}$ for some $e \in \mathcal{O}$, and that every well order is isomorphic to some $d \in \mathcal{O}$.

A **surreal** s is a function $s : d \rightarrow \{+, -\}$ for some $d \in \mathcal{O}$. For such a surreal, we call d the **domain** of it, and denote this as $D(s) = d$. It's clear that $s(e) = +$ or $-$ for $e \in d$; we will add the formal notation that $s(e) = \emptyset$ when $e \notin D(s)$.

Let \mathcal{S}_d denote the set of surreals $\{s : \|D(s)\| < \aleph_d\}$.

We'll use the notation $\{< d\}$ to indicate the set of ordinals less than d . Technically, this is equal to d itself, but we will need to refer to $s(\{< d\}) = \{s(e) : e < d\}$ as a sequence of signs, as opposed to $s(d)$, indicating a single sign.

Consider surreals s and t . If they have $D(s) = D(t)$ and $s(d) = t(d) \ \forall d \in D(s)$, then we write $s = t$. Otherwise, let d be the first value in $D(s) \cup D(t)$ where $s(d) \neq t(d)$; we call d the **first difference coordinate**. We use the convention that $- < \emptyset < +$, for these definitions: $s < t$ iff $s(d) < t(d)$ and $s > t$ iff $s(d) > t(d)$. This can be considered a lexicographical order, and hence is a linear order on the surreals.

Given two orders X and Y , we say that X can be **order-embedded** in Y iff $\exists f : X \rightarrow Y$ with $x < y \implies f(x) < f(y)$; we call such f an **order-embedding**.

3. MAIN RESULTS

Theorem 1. *Suppose \aleph_d is either a successor cardinal, or is \aleph_0 . Suppose e, f are ordinals of cardinality less than \aleph_d , and that $(s_i)_{i \in e} \subset S_d$ is increasing, $(t_j)_{j \in f} \subset S_d$ is decreasing, and $s_i < t_j \ \forall i, j$. Then there exists an element $r \in S_d$ such that $s_i < r < t_j \ \forall i, j$.*

Theorem 2 (Universal Embedding). *If X is a linear order with $\|X\| \leq \aleph_d$, then X can be order-embedded in S_d .*

4. PROOFS

We will define a very useful limit idea on surreals that will eventually give us the proof of theorem 1. From there, it is easy to get to theorem 2.

4.1. The Surreal Limit. Suppose we have an increasing sequence of surreals $(s_i)_{i \in e}$ indexed by some ordinal e . Then we define the **surreal limit** s of (s_i) , written as $s = \text{slim}_{i \in e} s_i$, like this:

$$s(d) = \begin{cases} \lim_{i \in e} s_i(d) & \text{if } \exists k \in e : i, j \geq k \implies s_i(\{< d\}) = s_j(\{< d\}) \\ \emptyset & \text{otherwise.} \end{cases}$$

We must check that this definition makes sense. There are two things to check.

First, we need to see that $\lim s_i(d)$ exists, as used in the definition. This clause of the definition is only relevant when $s_i(\{< d\})$ “settles” on to a specific sign sequence. From that point onward, the sign at $s_i(d)$ can only change in the order $(-, \emptyset, +)$ as i increases, since the sequence is increasing. Thus there must be some tail t of the index set e such that $s_i(d)$ is constant for $i \in t$. (For any ordinal d , a **tail** of d is a subset $t \subset d$ which is upward-closed in d ; in other words, $a \in t, b \in d, a < b \implies b \in t$.)

Second, we must check that if $s(d) = \emptyset$, then $s(f) = \emptyset$ for any $f > d$. Since $s(d) = \emptyset$, we know that $\forall k \in e \exists i, j \geq k : s_i(\{< d\}) \neq s_j(\{< d\})$. It follows immediately that this condition must also hold when $\{< d\}$ is replaced by any superset, including $\{< f\}$ for $f > d$. This completes the check that a surreal limit is well defined.

It is interesting to note that this limit exists for *any* increasing ordinal-indexed sequence, regardless of cardinality or boundedness.

Let’s note some basic properties of the surreal limit. Let $s = \text{slim } s_i$ and d be any ordinal in $D(s)$. Then each of the following is true:

$$(1) \quad \exists k \in e : \forall i \geq k, s_i(\{< d\}) = s(\{< d\}),$$

$$(2) \quad s_i(\{< d\}) = s(\{< d\}) \implies s(d) \geq s_i(d),$$

and

$$(3) \quad \lim_{i \in e} s_i(d) = s(d).$$

It is easy to verify each of these using the definition of s and the fact that s_i is increasing.

The surreal limit has some interesting properties which will lead us to the proof of theorem 1. In each of these properties, $(s_i)_{i \in e}$ is an increasing sequence of surreals, indexed by an ordinal e .

For surreals s, t , say that s is a **prefix** of t iff $D(s) = \{< d\} \subset D(t)$ and $t(\{< d\}) = s$; intuitively, this means t “starts with” s , as a sign sequence.

Property 3. *Any surreal t with prefix $s = \text{slim } s_i$ (including $s = t$) has $t \geq s_i \forall i \in e$.*

Proof. Suppose t has s as a prefix, and $t < s_i$ for some $i \in e$.

Let d be the first difference coordinate between t and s_i . If $d \in D(s)$, then $t(d) = s(d) \geq s_i(d)$ using property (2) above, contradicting that $t < s_i$.

So it must be the case that $d \notin D(s)$. This means that $s_i(D(s)) = s$, and in fact $s_j(D(s)) = s$ for any $j > i$; otherwise let $e \in D(s)$ be the first difference coordinate between $s_i < s_j$, and we would have a contradiction to property (2), setting $d = e$.

Let ordinal $f = D(s)$. By the definition of s , $s(f)$ must be defined (and $\neq \emptyset$) since $s_j(\{< f\}) = s_k(\{< f\}) \forall j, k > i$. This is another contradiction since we chose f such that $s(f) = \emptyset$. The conclusion is that it is impossible for $t < s_i$. \square

Note that if e is a limit ordinal and the s_i are strictly increasing, then this result also implies that $t > s_i$ for all $i \in e$, since if $t = s_i$ for any i , then we would get $t < s_{i+1}$, which is impossible.

Property 4. Suppose $t < s = \text{slim } s_i$, and s is not a prefix of t . Then $t < s_i$ for some $i \in e$.

Proof. Let d be the first difference coordinate between s and t . We know $d \in D(s)$; otherwise s would be a prefix of t . By property (1) there is a k such that $i \geq k \implies s_i(\{< d\}) = s(\{< d\}) = t(\{< d\})$. By property (3), there is an $i \geq k$ such that $s_i(d) = s(d) > t(d)$, which means $s_i > t$. \square

The next property is the essence of theorem 1. Now is a good time to note that the definition of a surreal limit works equally well for decreasing sequences, and that the relevant properties apply in those cases with their inequalities reversed. We'll also say that surreals s, t are **relatives** if either s is a prefix of t , or vice versa.

Property 5. Suppose $(s_i)_{i \in e}$ is an increasing sequence of surreals, $(t_i)_{i \in f}$ is decreasing, with $s_i < t_j \forall i \in e, j \in f$. Let $s = \text{slim } s_i$ and $t = \text{slim } t_j$. Then either $s < t$ or s, t are relatives.

Proof. If $s < t$ or $s = t$, we are done, so suppose $t < s$, and that they are not relatives. Then the first difference coordinate d has $t(d) < s(d)$. We know that $s(d) \neq \emptyset$ and $t(d) \neq \emptyset$; otherwise, they would be relatives. Thus $t(d) = -$ and $s(d) = +$. Define the surreal r to have domain $\{< d\}$, with $r(\{< d\}) = s(\{< d\}) = t(\{< d\})$, so that $t < r < s$, and neither s nor t is a prefix of r .

Then property 4 provides some $s_i > r$ and some $t_j < r$. But this would mean $t_j < s_i$, a contradiction. Therefore, it's impossible to have $t < s$ while s, t are not relatives. \square

The next property is not as elegant, but will be useful in proving theorem 1.

Property 6. If infinite cardinal C is a successor cardinal or \aleph_0 , ordinal e has $\|e\| < C$, and $\|D(s_i)\| < C \forall i \in e$, then $\|D(s)\| < C$ for $s = \text{slim } s_i$.

Proof. We have

$$D(s) \subset \max_{i \in e} D(s_i) \subset \sum D(s_i),$$

so $\|D(s)\| \leq \sum_{i \in e} \|D(s_i)\|$. If $C = \aleph_0$, then this is a finite sum of finite numbers, which is also clearly finite (i.e. $\|D(s)\| < C$), and we would be done.

Otherwise, suppose C^- is the immediate successor of C . Then $\|D(s_i)\| \leq C^-$ for all i , and $\sum \|D(s_i)\| \leq (C^-)^2 = C^-$, which completes the proof. \square

4.2. Proofs of the main results. The theorems are restated here for convenience.

Theorem 1. Suppose \aleph_d is either a successor cardinal, or is \aleph_0 . Suppose e, f are ordinals of cardinality less than \aleph_d , and that $(s_i)_{i \in e} \subset S_d$ is increasing, $(t_j)_{j \in f} \subset S_d$ is decreasing, and $s_i < t_j \forall i, j$. Then there exists an element $r \in S_d$ such that $s_i < r < t_j \forall i, j$.

Proof. Let $s = \text{slim } s_i$ and $t = \text{slim } t_j$. By property 6, both s and t are in S_d . By property 5, either $s < t$ or one is the prefix of the other.

If they are not relatives, then $s < t$, and by property 3, any surreal r in $[s, t]$ has $s_i < r < t_j \forall i, j$. We will construct a specific surreal $r \in S_d$ with $s < r < t$. In particular, let $d = D(s)$ and choose r with $r(\{< d\}) = s$ and $r(d) = +$, so that $s < r$. The first difference coordinate between s and t must be within $D(s)$ (otherwise they would be relatives), so we also have $r < t$. By property 6, $\|D(r)\| < \aleph_d$, so $r \in S_d$. Thus r satisfies the theorem when $s < t$.

Now suppose that s is a prefix of t , and let $r = t$. By property 3, $s_i < r$ and $r < t_j$ for all $i \in e, j \in f$. By property 6, $\|D(r)\| < \aleph_d$, so again r satisfies the conditions of the theorem. If t is a prefix of s , we can similarly set $r = s$. This shows that the required surreal r exists in all cases. \square

Theorem 2. If X is a linear order with $\|X\| \leq \aleph_d$, then X can be order-embedded in S_d .

Proof. The axiom of choice (AoC) is used in this proof.

Suppose X , of size \aleph_d , is linearly ordered by $<_t$, and we give it a well order $<_w$ such that $|\{<_w x\}| < \aleph_d \forall x \in X$. We will recursively define an order embedding $f : X \rightarrow S_d$, using transfinite induction on $<_w$.

Suppose f has been defined on $\{<_w x'\}$. Using x' , we will define sequences $s_i, t_j \in S_d$ such that the condition $s_i < f(x') < t_j$ is necessary and sufficient for f to remain an order embedding.

Recursively define $s_i \in S_d$ as an element (using AoC) of $\{f(x) : x <_w x', x <_t x', s_j < f(x) \forall j < i\}$, unless this set is empty, in which case the sequence s_i terminates. Similarly define $t_j \in S_d$ as an element of $\{f(x) : x <_w x', x' <_t x, f(x) < t_k \forall k < j\}$, until that set is empty. It's possible that either or both of these sequences may be empty, which does not cause any difficulty. Note that s_i is increasing, and t_j is decreasing, and $f^{-1}(s_i) <_t x' <_t f^{-1}(t_j) \forall i, j$, so that $s_i < t_j$.

If \aleph_d is a successor or \aleph_0 , we can apply theorem 1 to find an element $r \in S_d$ with $s_i < r < t_j$, and we set $f(x') = r$. By transfinite induction, this results in an order embedding f which satisfies the theorem in this case.

If $\aleph_d = \cup_{e < d} \aleph_e$, then we note that the above construction is capable of extending *any* order embedding $f : \{< x'\} \rightarrow S_d$ by adding $f(x')$. For ordinal e , let x_e denote the $<_w$ -first element of X with $|\{<_w x_e\}| = \aleph_e$. Start by using the above method to find $f_0 : \{<_w x_0\} \rightarrow S_0$. Then at each step extend this from $\cup_{i < e} f_i$ to $f_e : \{<_w x_e\} \rightarrow S_e$ using the above technique. (This can be done since $i < j \implies S_i \subset S_j$.) Let f be the union of all these functions; this is an order embedding satisfying the theorem. \square

To conclude, we note that the size of S_d is

$$||S_d|| = \sum_{e < d} 2^{\aleph_e}.$$

(This notation is assuming the cardinals can be well-ordered, which is a consequence of the axiom of choice.) Under the generalized continuum hypothesis, $2^{\aleph_e} = \aleph_{e+1}$, and it follows that $||S_d|| = \aleph_d$.

5. EDIT THIS PAPER

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